Problem Set 5 due April 8, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Invent an algorithm (explain all the steps in words, and explain why it works) which takes general vectors $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ as inputs, and decides whether $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. *(15 points)*

Problem 2:

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Let $V = N(A)$ be the nullspace of A, and $W = C(A^T)$ be the row space of A.

(1) Compute bases of V and W. *(10 points)*

(2) Compute the projection matrices P_V onto V and P_W onto W. *(10 points)*

(3) Compute $P_V + P_W$. The answer should be a very nice matrix. Explain geometrically why you get this answer (*Hint: it has to do with the geometric relation between V and W*). *(5 points)*

Problem 3:

Consider the following lines L_1 and L_2 in 3-dimensional space:

$$L_1 = \left\{ \begin{bmatrix} x+1 \\ x \\ x \end{bmatrix} \text{ for } x \in \mathbb{R} \right\} \qquad \text{and} \qquad L_2 = \left\{ \begin{bmatrix} y \\ 2y \\ 3y \end{bmatrix} \text{ for } y \in \mathbb{R} \right\}$$

(1) Which of these is a subspace and which is not? Explain why. *(5 points)*

(2) Consider a point $R = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ on the line which **IS NOT** a subspace among L_1 and L_2 (so if R is on L_1, the entries a, b, c are given in terms of x, while if R is on L_2, they are given in terms of y). Compute the smallest possible distance from R to the other line among L_1 and L_2. *(5 points)*

(3) By minimizing the quantity in part (2), find the points $P \in L_1$ and $Q \in L_2$ for which the distance $|PQ|$ is minimal among all possible choices of a point on either line. *(5 points)*

(4) What can you say about the line PQ in relation to the lines L_1 and L_2? *(5 points)*

Problem 4:

Consider the following cubic curve in the xy plane: $y = ax^3 + bx + c$.

(1) Compute a, b, c such that the curve passes through the points $(0, 1)$, $(1, 0)$ and $(2, 5)$ (don't just guess, use linear algebra to solve for a, b, c). *(10 points)*

(2) Compute a, b, c such that the cubic curve is the best fit for the points $(0, 1)$, $(1, 0)$, $(2, 5)$ and $(3, -1)$: this means that the sum of the squares of the vertical distances between the curve and the four given points should be minimum (*Hint: this is done similarly to the example of fitting a line, that we did at the end of Lecture 13*). *(10 points)*

Problem 5:

(1) Use Gram-Schmidt to compute an orthonormal basis of $\mathbb{R}^4$ that includes the vector:

$$\boldsymbol{q}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(10 points)

(2) Compute the $A = QR$ factorization of the matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -5 & -1 \\ 2 & -5 & -3 \end{bmatrix}$$

(where Q has orthonormal columns and R is square upper triangular). *(10 points)*